Geometric conditions which imply compactness of the $\bar{\partial}$-Neumann operator*

Emil J. Straube
Department of Mathematics, Texas A&M University, College Station,TX 77843
straube@math.tamu.edu

Abstract

For smooth bounded pseudoconvex domains in $\mathbb{C}^2$, we provide geometric conditions on (the points of infinite type in) the boundary which imply compactness of the $\bar{\partial}$-Neumann operator. It is noteworthy that the proof of compactness does *not* proceed via verifying the known potential theoretic sufficient conditions.

Let Ω be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$. The $\bar{\partial}$-Neumann operator N on (0,1)-forms is the inverse of the complex Laplacian $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. N and its regularity theory play a crucial role both in partial differential equations and in several complex variables ([10], [2], [6]). In particular, the question when N is a compact operator on $\mathcal{L}^2_{(0,1)}(\Omega)$, the space of (0,1)-forms with square integrable coefficients, is of interest in several contexts. Well known examples are global regularity and the Fredholm theory of Toeplitz operators. We refer the reader to [12] for more information and for references on these and other questions related to compactness of the $\overline{\partial}$-Neumann operator. Among more recent references, we mention [14], where compactness is related to the existence of Henkin–Ramirez type integral formulas with well behaved kernels. And, there is a useful connection between compactness in the $\overline{\partial}$-Neumann problem and the asymptotic behavior, in a semi–classical limit, of the ground state energy of certain magnetic Schrödinger operators ([13], [7]).

The most general known sufficient condition for compactness is potential theoretic in nature. Roughly speaking, there should exist, near the boundary points of infinite type, plurisubharmonic functions with arbitrarily large complex Hessians whose gradients are uniformly bounded in the metric induced by the Hessians of the functions. This condition was introduced under the name property $(\widetilde{P})$ and shown to imply compactness by McNeal in [16]. It generalizes previous work by Catlin that used his now classical condition property (P) ([4], see [17] for a systematic study of this property). Property $(\widetilde{P})$ implies compactness of the $\overline{\partial}$-Neumann operator on an arbitrary bounded

*Research supported in part by NSF grant DMS 0100517

pseudoconvex domain, even when no boundary regularity whatsoever is assumed ([18], Corollary 3; [16], Corollary 4.2).

There are natural geometric conditions, bearing on how the set of infinite type boundary points sits inside the boundary, which are known to imply property (P) or $(\widetilde{P})$ and hence compactness of the $\overline{\partial}$-Neumann operator ([4], [17], [1], [12], [11], [16]). In this paper, we present new geometric conditions which imply compactness in the case of smooth bounded pseudoconvex domains in $\mathbb{C}^2$. In light of the discussion in the next paragraph, the fact that compactness is *not* demonstrated via establishing property (P) or $(\widetilde{P})$ (in contrast to ([4], [17], [1], [12], [11], [16]) is worth pointing out.

On domains which are locally convexifiable, the analysis, the potential theory, and the geometry associated with the $\overline{\partial}$-Neumann problem mesh perfectly. That is, the following three statements are equivalent: (i) the $\overline{\partial}$-Neumann operator is compact, (ii) the boundary of the domain satisfies property $(\widetilde{P})$, (iii) the boundary of the domain does not contain (germs of) analytic discs ([11], [12], Theorem 5.1). This is far from true in general. Property$(\widetilde{P})$ always excludes discs from the boundary (as is seen by pulling back the good plurisubharmonic functions to the unit disc in the complex plane); so does compactness of N for domains in $\mathbb{C}^2$ with at least Lipschitz boundary (see [12], Proposition 4.1 for a proof of this folklore result). However, Sibony ([17]) observed that the absence of discs from the boundary need not imply property $(\widetilde{P})$, and Matheos ([15], see also [12]) showed that in fact the absence of discs need not (even) imply compactness of the $\overline{\partial}$-Neumann operator. (Actually, Sibony's observation concerned property (P), but the domains in [17] are Hartogs domains in $\mathbb{C}^2$ where $(\widetilde{P})$ and (P) are equivalent ([13], Appendix A).) This left open the exact relationship between property $(\widetilde{P})$ and compactness of the $\overline{\partial}$-Neumann operator. Christ and Fu ([7]) recently established the equivalence of these two properties on smooth bounded pseudoconvex Hartogs domains in $\mathbb{C}^2$. On general domains, however, the situation is not understood. In view of this, it is desirable to have a technique like ours for establishing compactness of the $\overline{\partial}$-Neumann operator that does not rely on property $(\widetilde{P})$ (in this context, compare also [14]).

If Z is a (real) vector field defined on some open subset of $b\Omega$ (or of $\mathbb{C}^2$), we denote by $\mathcal{F}^t_Z$ the flow generated by Z. By finite type of a boundary point, we mean finite type in the sense of D'Angelo; however, for domains in $\mathbb{C}^2$, the various notions of finite type all agree, so that no distinction is necessary ([8]). Recall that the set of points of finite type in the boundary of a smooth bounded pseudoconvex domain is open and, consequently, the set of points of infinite type is compact. $B(P,r)$ denotes the open ball of radius r centered at P.

Theorem. *Let Ω be a C^∞-smooth bounded pseudoconvex domain in $\mathbb{C}^2$. Denote by K the set of boundary points of infinite type. Assume that there exist constants $C_1, C_2 > 0$, C_3 with $1 \leq C_3 < 3/2$, and a sequence $\{\varepsilon_j > 0\}_{j=1}^\infty$ with $\lim_{j\to\infty} \varepsilon_j = 0$ so that the following holds. For every $j \in \mathbb{N}$ and $P \in K$ there is a (real) complex tangential vector field $Z_{P,j}$ of unit length defined in some neighborhood of P in $b\Omega$ with $\max|div\ Z_{P,j}| \leq C_1$ such that $\mathcal{F}^{\varepsilon_j}_{Z_{P,j}}(B(P, C_2(\varepsilon_j)^{C_3}) \cap K) \subseteq b\Omega \backslash K$. Then the $\overline{\partial}$-Neumann operator on Ω is compact.*

It is of course assumed that the flow $\mathcal{F}^t_{Z_{P,j}}$ exists for all initial points in $B(P, C_2(\varepsilon_j)^{C_3}) \cap K$ and $t \leq \varepsilon_j$.

Two immediate questions arise. First, the obvious examples that satisfy the assumptions in the theorem also satisfy property($\widetilde{P}$), and whether or not the theorem can actually furnish examples (if they exist) of domains with compact $\overline{\partial}$-Neumann operator, but without property ($\widetilde{P}$), does not appear to be a simple matter to decide. But we obtain, in any event, a simple geometric proof of compactness under the conditions in the theorem. Moreover, these conditions turn out to be natural and, modulo the form of the lower bound $C_2(\varepsilon_j)^{C_3}$, minimal; we discuss this in Remark 3 below. Second, our result is stated for domains in $\mathbb{C}^2$; the proof uses maximal estimates and so does not work in higher dimensions. However, it is easy to see that the theorem, when transcribed to $\mathbb{C}^n$ verbatim, also fails. Control over K will have to be imposed in all complex tangential directions rather than just in one. We hope to return to these questions elsewhere.

The general thrust in the theorem is that at points of K there should exist a (real) complex tangential direction transversal to K in which $b\Omega\backslash K$ (the good set) is thick enough. This occurs in extremis when K is (locally) contained in a totally real submanifold of the boundary (property(P), and hence compactness, are well known in this case ([4])). The following corollary to the theorem takes this situation and disposes of the requirement that K be (contained in) a smooth submanifold of the boundary.

Corollary 1. *Let Ω be a C^∞-smooth bounded pseudoconvex domain in $\mathbb{C}^2$. Denote by K the set of boundary points of infinite type. Assume that for all $P \in K$ there exists a (real) complex tangential vector field Z_p defined near P, a neighborhood U_p of P, and $\varepsilon_p > 0$ such that $\mathcal{F}^t_{Z_p}(U_p \cap K) \subseteq b\Omega\backslash K$ for $t \leq \varepsilon_p$. Then the $\overline{\partial}$-Neumann operator on Ω is compact.*

To reduce Corollary 1 to the theorem, it suffices to cover K by finitely many neighborhoods U_P.

A second geometrically simple special case occurs when $b\Omega\backslash K$ satisfies what might be called a weak complex tangential cone condition. That is, there should exist a finite (possibly small) open real cone C in $\mathbb{C}^2 \approx \mathbb{R}^4$ having the following property. For each $P \in K$ there exist a complex tangential direction so that when C is moved by a rigid motion to have vertex at P and axis in that complex tangential direction, the (open) cone obtained intersects $b\Omega$ in a set contained in $b\Omega\backslash K$. Then the assumptions in the theorem are satisfied with $C_3 = 1$.

Corollary 2. *Let Ω be a C^∞-smooth bounded pseudoconvex domain in $\mathbb{C}^2$. Denote by K the set of boundary points of infinite type. Assume that $b\Omega\backslash K$ satisfies a weak complex tangential cone condition. Then the $\overline{\partial}$-Neumann operator on Ω is compact.*

Because C_3 is only required to be in the range $1 \leq C_3 < 3/2$, the cone in Corollary 2 may be allowed to degenerate into a mild cusp; we leave the details to the reader.

Remark 1. The assumption in the theorem says that for all $j \in \mathbb{N}$, all $P \in K$, there should exist a complex tangential vector field and a ball centered at P so that the intersection of K with this ball, when translated along the integral curves of the vector field by ε_j, ends up in $b\Omega\backslash K$, i.e. in the set of points of finite type. The crucial additional condition is that the radius of this ball should be at

least of order $(\varepsilon_j)^{C_3}$ for some $C_3 < 3/2$. A condition of this kind is needed; it is not enough to just have some ball centered at P. Consider a Hartogs domain $\Omega := \{(z,w) \in \mathbb{C}^2 / |z| < 1, |w| < e^{-\varphi(z)}\}$, where φ is smooth and plurisubharmonic on the unit disc, and such that Ω is a smooth domain in $\mathbb{C}^2$, which is strictly pseudoconvex at boundary points where $w = 0$. Let $K_0 := \{z \in D \mid \Delta\varphi = 0\}$ be relatively compact in D. Assume that $\Delta\varphi$ vanishes to infinite order on K_0. It is well known that the set K of boundary points of infinite type consists precisely of those $(z,w) \in b\Omega$ with $z \in K_0$. Finally, assume that K_0 has empty interior (this is equivalent to $b\Omega$ not containing analytic discs). If $P = (z,w) \in K \subseteq b\Omega$, then $z \in K_0$. Because K_0 has empty interior, there are arbitrarily short translates of z in $D \backslash K_0$. Because $D \backslash K_0$ is open, there exists for every $\varepsilon > 0$ a vector X and a ball $B(z,r)$ such that $B(z+X,r) \subseteq D \backslash K_0$. The (constant) vector field X is easily lifted to a complex tangential vector field on $b\Omega$ (near P). In this way, one sees that Ω satisfies the assumptions of the theorem, except for the lower bound (in terms of ε_j) of the balls centered at points of K. This is enough to make the theorem fail: there are examples of Hartogs domains as above, where the $\bar{\partial}$-Neumann operator is not compact ([15], [12], Theorem 4.2 and Remark 5).

Remark 2. The key to getting the vector fields from the theorem in the examples in Remark 1 is that K_0 has empty interior, or equivalently, that $b\Omega$ contains no analytic discs. It turns out that suitable vector fields may be obtained on any domain in $\mathbb{C}^2$ whose boundary contains no analytic disc. In fact, the boundary of a smooth bounded pseudoconvex domain in $\mathbb{C}^2$ contains no analytic discs if and only if it satisfies all the assumptions of the theorem, except possibly the lower bounds (in terms of ε_j) on the radii of the balls centered at points $P \in K$. It is clear that the assumptions in the theorem, without these lower bounds, exclude discs from the boundary. The proof of the other direction follows easily from Proposition 3.1.12 in ([3]), as follows. Let $P \in b\Omega$, Z a real vector field defined on $b\Omega$ in a neighborhood of P that is complex tangential, and let J denote the involution associated with the complex structure of $\mathbb{C}^n$. Then Z and $J(Z)$ span (over $\mathbb{R}$) the complex tangent space to $b\Omega$ near P. For $0 \leq \theta < 2\pi$, denote by Z^θ the field $Z^\theta := (\cos\theta)Z + (\sin(\theta)J(Z)$. Fix $\varepsilon > 0$. Catlin shows in [3], Proposition 3.1.12 that if all points of $M_\varepsilon := \{\mathcal{F}^t_{Z^\theta}(P) \mid 0 \leq t < \varepsilon, 0 \leq \theta < 2\pi\}$ are weakly pseudoconvex points of $b\Omega$, then M_ε is a (necessarily one-complex dimensional) complex manifold. Consequently, when there is no analytic disc in $b\Omega$, there exist complex tangential fields $Z_{P,\varepsilon}$ for all $P \in K$ and sufficiently small $\varepsilon > 0$ so that $\mathcal{F}^\varepsilon_{Z_{P,\varepsilon}}(z) \notin K$ for z close enough to P. Moreover, the $Z_{P,\varepsilon}$ are of the form $Z_{P,\varepsilon} = (\cos\theta_{P,\varepsilon})Z + (\sin\theta_{P,\varepsilon})J(Z)$, so that div $Z_{P,\varepsilon}$ is bounded uniformly in P and ε.

Remark 3. Since on a smooth bounded pseudoconvex domain in $\mathbb{C}^2$, compactness of the $\overline{\partial}$-Neumann operator implies that there are no discs in the boundary (see the discussion preceeding the statement of the theorem), we can deduce from Remark 2 a (very) partial converse to the theorem. If N is compact, then the domain satisfies the assumptions in the theorem except possibly the lower bound $C_2(\varepsilon_j)^{C_3}$ on the radii of the balls whose intersections with K, when translated along the integral curves of the fields $Z_{P,j}$, end up in $b\Omega \backslash K$. Moreover, the examples in Remark 1 show that there must be some lower bound on the size of these balls in terms of ε_j, otherwise the theorem need not hold. Thus the assumptions in the theorem are quite natural, and, modulo the exact form of this

lower bound, minimal, as asserted in the discussion following the statement of the theorem.

We now prove the theorem. There are various equivalent statements to compactness of the $\bar{\partial}$-Neumann operator, see e.g. [12], Lemma 1.1. We will show a so-called compactness estimate, that is, we will show that for all $\varepsilon > 0$ there exists a constant C_ε such that

$$\|u\|_0^2 \leq \varepsilon(\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2) + C_\varepsilon\|u\|_{-1}^2, \qquad u \in \mathrm{dom}(\bar{\partial}^*) \cap C^\infty_{(0,1)}(\overline{\Omega}). \tag{1}$$

Here, $C^\infty_{(0,1)}(\overline{\Omega})$ denotes the space of (0,1)-forms with coefficients in $C^\infty(\overline{\Omega})$, the norm $\| \ \|_0$ and $\| \ \|_{-1}$ denote the Sobolev-0 (i.e. $\mathcal{L}^2(\Omega)$) and Sobolev-1 norms, respectively. For forms, these norms are computed componentwise. Note that if (1) holds for $u \in \mathrm{dom}(\bar{\partial}^*) \cap C^\infty_{(0,1)}(\overline{\Omega})$, it holds for $u \in \mathrm{dom}(\bar{\partial}^*) \cap \mathrm{dom}(\bar{\partial})$, because the former space is dense in the latter with respect to the graph norm (see for example [6], Lemma 4.3.2).

The idea of the proof is very simple. To estimate the $\mathcal{L}^2$-norm of u near a point P of K, we express u there in terms of u in a patch which meets the boundary in a relatively compact subset of the set of finite type points plus the integral of the derivative of u in the direction $Z_{P,j}$. The first contribution is easily handled by subelliptic estimates, while the second is estimated by the length of the curve (which is ε_j) times the $\mathcal{L}^2$-norm of $Z_{P,j}u$. But in $\mathbb{C}^2$, this $\mathcal{L}^2$-norm is estimated by the $\mathcal{L}^2$-norm of $\bar{\partial}u$ and $\bar{\partial}^*u$, because $Z_{P,j}$ is complex tangential. Controlling the terms coming from the integral of $Z_{P,j}u$ raises overlap and divergence issues; these are taken care of by the uniformity built into the assumption in the theorem.

The details are as follows. First note that we can extend the fields $Z_{P,j}$ form $b\Omega$ to the inside of Ω by a fixed distance by letting them be constant along the real normal. We still denote these extended fields by $Z_{P,j}$; they are complex tangential to the level sets of the boundary distance.

Fix $\varepsilon > 0$ and j so that $\varepsilon_j < \varepsilon$. A standard covering theorem (see e.g. [19], Theorem 1.3.1) applied to the family of closed balls $\{\overline{B(P, \frac{C_2}{10}(\varepsilon_j)^{C_3})} \mid P \in K\}$ gives a subfamily of pairwise disjoint balls so that the corresponding closed balls of radius $\frac{C_2}{2}(\varepsilon_j)^{C_3}$, hence the open balls of radius $C_2(\varepsilon_j)^{C_3}$, still cover K. Because K is compact, we obtain a finite family of open balls $\{B(P_k, C_2(\varepsilon_j)^{C_3}) \mid 1 \leq k \leq N, P_k \in K\}$ that covers K, and such that the corresponding closed balls of radius $\frac{C_2}{10}(\varepsilon_j)^{C_3}$ are pairwise disjoint. To simplify notation, we will use Z_k to denote the fields $Z_{P_k,j}$, $1 \leq k \leq N$. By decreasing C_2 in the theorem, we may assume that $\mathcal{F}^{\varepsilon_j}_{Z_k}(B(P_k, C_2(\varepsilon_j)^{C_3}) \cap K)$ is not only contained in $b\Omega \backslash K$, but is relatively compact there. Consequently there exist open subsets U_k, $1 \leq k \leq N$, of $\overline{\Omega}$, with

$$K \cap B(P_k, C_2(\varepsilon_j)^{C_3}) \subseteq U_k \subseteq B(P_k, C_2(\varepsilon_j)^{C_3}) \tag{2}$$

and

$$\mathcal{F}^{\varepsilon_j}_{Z_k}(U_k \cap \Omega) \cap K = \emptyset. \tag{3}$$

It is implicit in the statement of (3) that U_k is chosen in a sufficiently small neighborhood of $b\Omega$ so that the flow generated by the extended vector fields Z_k exists up to time ε_j for all initial points in U_k.

Now let $u \in C^{\infty}_{(0,1)}(\overline{\Omega}) \cap \operatorname{dom}(\bar{\partial}^*)$. Then

$$\|u\|_0^2 = \int\limits_{\left(\bigcup\limits_{k=1}^{N} U_k\right)\cap\Omega} |u|^2 + \int\limits_{\Omega\setminus\left(\bigcup\limits_{k=1}^{N} U_K\right)} |u|^2. \tag{4}$$

Because $\Omega\setminus\left(\bigcup\limits_{k=1}^{N} U_k\right)$ meets $b\Omega$ in a compact subset of $b\Omega$ which does not intersect K, we can apply subelliptic estimates (see [5]) to estimate the second term in the right-hand side of (4): there exist $s > 0$ and $C > 0$ such that the restriction of u to a neighborhood U of $\Omega\setminus\left(\bigcup\limits_{k=1}^{N} U_k\right)$ (in Ω) belongs to $W^s_{(0,1)}(U)$ and

$$\|u\|^2_{W^s_{(0,1)}(U)} \leq C(\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2). \tag{5}$$

The usual interpolation inequality for Sobolev norms gives

$$\begin{aligned}\int\limits_{\Omega\setminus\left(\bigcup\limits_{k=1}^{n} U_k\right)} |u|^2 &\leq \|u\|_{\mathcal{L}^2_{(0,1)}(U)} \\ &\leq \frac{\varepsilon}{C}\|u\|^2_{W^s_{(0,1)}(U)} + C_\varepsilon\|u\|^2_{W^{-1}_{(0,1)}(U)} \\ &\leq \varepsilon(\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2) + C_\varepsilon\|u\|^2_{-1}.\end{aligned} \tag{6}$$

We now estimate the first term on the right-hand side of (4). Fix k, $1 \leq k \leq N$.

$$\begin{aligned}\int\limits_{U_k\cap\Omega} |u|^2 &= \int\limits_{U_k\cap\Omega} \left| u(\mathcal{F}^{\varepsilon_j}_{Z_k}(x)) - \int_0^{\varepsilon_j} Z_k u(\mathcal{F}^t_{Z_k}(x))dt \right|^2 dV(x) \\ &\leq 2\int\limits_{U_k\cap\Omega} |u(\mathcal{F}^{\varepsilon_j}_{Z_k}(x))|^2 dV(x) \\ &\quad + 2\int\limits_{U_k\cap\Omega} \left| \int_0^{\varepsilon_j} Z_k u(\mathcal{F}^t_{Z_k}(x))dt \right|^2 dV(x).\end{aligned} \tag{7}$$

The first term on the right-hand side of (7) can be estimated as follows:

$$\begin{aligned}\int\limits_{U_k\cap\Omega} |u(\mathcal{F}^{\varepsilon_j}_{Z_k}(x))|^2 dV(x) &= \int\limits_{\mathcal{F}^{\varepsilon_j}_{Z_k}(U_k\cap\Omega)} |u(y)|^2 \det(\partial x/\partial y) dV(y) \\ &\leq C_k \int\limits_{\mathcal{F}^{\varepsilon_j}_{Z_k}(U_k\cap\Omega)} |u(y)|^2 dV(y).\end{aligned} \tag{8}$$

Here we use $\det(\partial x/\partial y)$ as shorthand for the (positive) Jacobian of the diffeomorphism $\mathcal{F}_{Z_k}^{-\varepsilon_j}$: $\mathcal{F}_{Z_k}^{\varepsilon_j}(U_k \cap \Omega) \to U_k \cap \Omega$. By (3), we can use subelliptic estimates once more to estimate the last term in (8); an argument analogous to the one that led to (6) gives

$$\int\limits_{\mathcal{F}_{Z_k}^{\varepsilon_j}(U_k\cap\Omega)} |u(y)|^2 dV(y) \leq \frac{\varepsilon}{C_k N}(\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^* u\|_0^2) + C_\varepsilon \|u\|_{-1}^2. \tag{9}$$

We now estimate the second term on the right-hand side of (7). The Cauchy–Schwarz inequality and Fubini's theorem give

$$\begin{aligned}
\int\limits_{U_k\cap\Omega} \left| \int\limits_0^{\varepsilon_j} Z_k u(\mathcal{F}_{Z_k}^t(x))dt \right|^2 dV(x) &\leq \varepsilon_j \int\limits_{U_k\cap\Omega} \int\limits_0^{\varepsilon_j} |Z_k u(\mathcal{F}_{Z_k}^t(x))|^2 dt\ dV(x) \\
&= \varepsilon_j \int\limits_0^{\varepsilon_j} \int\limits_{U_k\cap\Omega} |Z_k u(\mathcal{F}_{Z_k}^t(x))|^2 dV(x)\ dt \\
&= \varepsilon_j \int\limits_0^{\varepsilon_j} \int\limits_{\mathcal{F}_{Z_k}^t(U_k\cap\Omega)} |Z_k u(y)|^2 \det(\partial x/\partial y) dV(y)\ dt \\
&\leq 2\varepsilon_j \int\limits_0^{\varepsilon_j} \int\limits_{\mathcal{F}_{Z_k}^t(U_k\cap\Omega)} |Z_k u(y)|^2 dV(y)\ dt.
\end{aligned} \tag{10}$$

In the last inequality in (10) we have used the uniform bound on the divergence of the fields Z_k, that is, on the rate of change of the volume element under the flows generated by the Z_k's. This bound implies that $\det(\partial x/\partial y) \leq \exp(tC_1) \leq \exp(\varepsilon_j C_1) \leq \exp(\varepsilon C_1) \leq 2$ for ε small enough. (It suffices to establish (1) for small enough $\varepsilon > 0$.)

Putting together estimates (7)–(10) and adding over k, we estimate the first term on the right-hand side of(4):

$$\begin{aligned}
\int\limits_{\left(\bigcup\limits_{k=1}^N U_k\right)\cap\Omega} |u|^2 &\leq \sum_{k=1}^N \int\limits_{U_k\cap\Omega} |u|^2 \\
&\leq \sum_{k=1}^N \left[\frac{2\varepsilon}{N}(\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^* u\|_0^2) + C_\varepsilon \|u\|_{-1}^2 \right. \\
&\qquad \left. + 4\varepsilon_j \int\limits_0^{\varepsilon_j} \int\limits_{\mathcal{F}_{Z_k}^t(U_k\cap\Omega)} |Z_k u(y)|^2 dV(y)\ dt \right].
\end{aligned} \tag{11}$$

That is,

$$\int\limits_{\left(\bigcup\limits_{k=1}^{N} U_k\right)\cap\Omega} |u|^2 \leq 2\varepsilon(\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2) + C_\varepsilon\|u\|_{-1}^2$$
$$+ 4\varepsilon_j \int\limits_0^{\varepsilon_j} \left(\sum_{k=1}^{N} \int\limits_{\mathcal{F}^t_{Z_k}(U_k\cap\Omega)} |Z_k u(y)|^2 dV(y) \right) dt\ . \tag{12}$$

Note that we adopt the usual convention of denoting by C_ε constants which depend only on ε, but whose actual value may change as the estimates progress.

No point of Ω is contained in more than $C(C_2)(\varepsilon_j)^{4-4C_3}$ of the sets $\mathcal{F}^t_{Z_k}(U_k\cap\Omega)$, $1 \leq k \leq N$, where $C(C_2)$ denotes a constant depending only on C_2. Indeed, if $Q \in \mathcal{F}^t_{Z_k}(U_k\cap\Omega)\cap\mathcal{F}^t_{Z_m}(U_m\cap\Omega)$, then by the triangle inequality, the distance between $\mathcal{F}^{-t}_{Z_k}(Q)$ and $\mathcal{F}^{-t}_{Z_m}(Q)$ is no more than $2t \leq 2\varepsilon_j$ (recall that $|Z_k| = 1$). Consequently, $B(P_m, \frac{C_2}{10}(\varepsilon_j)^{C_3}) \subseteq B(P_k, 2\varepsilon_j + 2C_2(\varepsilon_j)^{C_3} + \frac{C_2}{10}(\varepsilon_j)^{C_3})$. Since the balls $B(P_m, \frac{C_2}{10}(\varepsilon_j)^{C_3})$, $1 \leq m \leq N$, are pairwise disjoint, comparison of volumes gives the desired upper bound on how many of them can be contained in $B(P_k, 2\varepsilon_j + 2C_2(\varepsilon_j)^{C_3} + \frac{C_2}{10}(\varepsilon_j)^{C_3})$.

With this control over the overlap of the sets $\mathcal{F}^t_{Z_k}(U_k\cap\Omega)$, $1 \leq k \leq N$, for t fixed, we can control the last term in (12). Denote by L the complex tangential field of type (1,0); we may take L to be C^∞ on $\overline{\Omega}$, and so that $|\mathrm{Re}\ L|$ and $|\mathrm{Im}\ L|$ are one in a neighborhood of the boundary that contains all the sets $U_k\cap\Omega$, $1 \leq k \leq N$. Then, because $|Z_k| = 1$,

$$|Z_k u(y)|^2 \leq 2|(\mathrm{Re}\ L)u(y)|^2 + 2|(\mathrm{Im}\ L)u(y)|^2. \tag{13}$$

Inserting (13) into (12) and using the estimate on the overlap of the sets $\mathcal{F}^t_{Z_k}(U_k\cap\Omega)$, we obtain

$$\int\limits_{\left(\bigcup\limits_{k=1}^{N} U_k\right)\cap\Omega} |u|^2 \leq 2\varepsilon(\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2) + C_\varepsilon\|u\|_{-1}^2$$
$$+ 4\varepsilon_j \int\limits_0^{\varepsilon_j} 2C(C_2)(\varepsilon_j)^{4-4C_3} \int\limits_\Omega (|(\mathrm{Re}\ L)u(y)|^2 + |(\mathrm{Im}\ L)u(y)|^2) dV(y)\ dt \tag{14}$$
$$= 2\varepsilon(\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2) + C_\varepsilon\|u\|_{-1}^2$$
$$+ 8C(C_2)\varepsilon_j^{6-4C_3} \int\limits_\Omega (|(\mathrm{Re}\ L)u(y)|^2 + |(\mathrm{Im}\ L)u(y)|^2) dV(y).$$

Finally, we exploit maximal estimates ([9]): since we are in $\mathbb{C}^2$, the integral in the last term in (14) is dominated by $\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^*u\|_0^2$. Combining this with (4) and (6) shows that there is a constant

C independent of ε such that for all sufficiently small $\varepsilon > 0$, we have the estimate

$$\|u\|_0^2 \leq C(\varepsilon + \varepsilon^{6-4C_3})(\|\bar{\partial}u\|_0^2 + \|\bar{\partial}^* u\|_0^2) + C_\varepsilon \|u\|_{-1}^2. \tag{15}$$

(15) gives the required compactness estimate (1) (since C does not depend on ε and $6 - 4C_3 > 0$). This completes the proof of the theorem.

References:

1. Harold P. Boas, Small sets of infinite type are benign for the $\bar{\partial}$-Neumann problem, Proceedings of the American Math. Soc. **103**, 1988, pp. 569–578.
2. Harold P. Boas and Emil J. Straube, Global regularity of the $\bar{\partial}$-Neumann problem: a survey of the $\mathcal{L}^2$-Sobolev theory, *Several Complex Variables*, M. Schneider and Y.-T. Siu, eds., MSRI Publications **37**, Cambridge University Press, 1999.
3. David W. Catlin, Boundary behavior of holomorphic functions on weakly pseudoconvex domains, Princeton University Ph.D. Thesis, 1978.
4. David W. Catlin, Global regularity of the $\bar{\partial}$-Neumann problem, *Complex Analysis of Several Variables*, Y.-T. Siu ed., Proc. Symp. Pure Math. **41**, 1984, pp. 39–49.
5. David W. Catlin, Subelliptic estimates for the $\bar{\partial}$-Neumann problem on pseudoconvex domains, Annals of Mathematics (2) **126**, 1987, pp. 131–191.
6. So-Chin Chen and Mei-Chi Shaw, *Partial Differential Equations in Several Complex Variables*, Studies in Advanced Mathematics, American Mathematical Society/International Press, 2001.
7. Michael Christ and Siqi Fu, Compactness in the $\bar{\partial}$-Neumann problem, magnetic Schrödinger operators, and the Aharonov–Bohm effect, preprint 2003.
8. John P. D'Angelo, *Several Complex Variables and the Geometry of Real Hypersurfaces*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1993.
9. M. Derridj, Regularité pour $\bar{\partial}$ dans quelques domaines faiblement pseudo-convexes, Journal of Differential Geometry **13**, 1978, pp. 559–576.
10. G.B. Folland and J.J. Kohn, *The Neumann Problem for the Cauchy–Riemann Complex*, Annals of Mathematics Studies **75**, Princeton University Press, 1972.
11. Siqi Fu and Emil J. Straube, Compactness of the $\bar{\partial}$-Neumann problem on convex domains, Journal of Functional Analysis **159**, 1998, pp. 629–641.
12. Siqi Fu and Emil J. Straube, Compactness in the $\bar{\partial}$-Neumann problem, *Complex Analysis and Geometry*, J. McNeal ed., Ohio State Univ. Math. Research Inst. Publ. **9**, 2001, pp. 141–160.

13. Siqi Fu and Emil J. Straube, Semi-classical analysis of Schrödinger operators and compactness in the $\bar{\partial}$-Neumann problem, Journal of Math. Analysis and Application **271**, 2002, pp. 267–282, correction in Journal of Math. Analysis and Applications **280**, 2003, pp. 195–196.

14. Torsten Hefer and Ingo Lieb, On the compactness of the $\overline{\partial}$-Neumann operator, Ann. Fac. Sci. Toulouse Math.(6) **9**, 2000, pp. 415–432.

15. Peter Matheos, A Hartogs domain with no analytic discs in the boundary for which the $\bar{\partial}$-Neumann problem is not compact, University of California Los Angeles Ph.D. Thesis, 1997.

16. Jeffery D. McNeal, A sufficient condition for compactness of the $\bar{\partial}$-Neumann problem, Journal of Functional Analysis **195**, 2002, pp. 190–205.

17. Nessim Sibony, Une classe de domaines pseudoconvexes, Duke Math. Journal **55**, 1987, pp. 299–319.

18. Emil J. Straube, Plurisubharmonic functions and subellipticity of the $\overline{\partial}$-Neumann problem on non-smooth domains, Mathematical Research Letters **4**, 1997, pp. 459–467.

19. William P. Ziemer, *Weakly Differentiable Functions*, Graduate Texts in Mathematics, vol. 120, Springer-Verlag, 1989.